# Consecutive measurements of photon number and quantum phase

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**Abstract.** - We introduce the conditional probability to consider consecutive measurements of photon number and quantum phase of a single mode. Let $\mathcal{P}$ be the conditional probability to measure the phase $\alpha$ with precision $\Delta\alpha$, given a previous measurement of $k$ photons with precision $\Delta k$. Two upper bounds of the probability $\mathcal{P}$ are derived. For arbitrary given precisions $\Delta k$ and $\Delta\alpha$, these bounds refer to the variation of $k$, $\alpha$, and the state vector $\psi$ in Hilbert space. The first (weaker) bound is given by the inequality $\mathcal{P} \leq \xi$, with $\xi = \frac{\Delta\alpha\,(\Delta k+1)}{2\pi}$. It is nontrivial for measurements with $\xi < 1$. As our main result the *least upper bound* of $\mathcal{P}$ is determined. We obtain an analytical representation of this bound in the asymptotic limit $\Delta k \to \infty$ and $\Delta\alpha \to 0$ such that $\xi > 0$ is fixed. Finally, we present a rigorous prove that the well-known *Heisenberg limit* in precision phase measurement can never be attained with measurement probabilities greater than $1/\pi$.

The classical picture for the evolution of a single-mode electromagnetic field is simply determined by an amplitude (specifying the strength of the field) and a phase (specifying the zeros of the field). In quantum theory, the field strength may be specified exactly in terms of photon number $N$. On the other hand, the concept of electromagnetic phase as an observable quantity is a long-standing problem of quantum optics and it has been the question whether there exists a phase observable that is canonically conjugate to the number observable for a single-mode field. The quantum mechanical description of phase was first considered by London [1] and Dirac [2]. An obvious way of defining an operator for the phase is by polar decomposition of the photon annihilation operator $\hat{a} = e^{i\hat{\phi}}\sqrt{\hat{N}}$. The phase operator $\hat{\phi}$ defined in this way is equivalent to that considered by Dirac [2], who obtained the commutator $[\hat{\phi}, \hat{N}] = i$ by employing the correspondence between commutators and classical Poisson bracket. Formally, this would imply the uncertainty relation

$$\sigma_\phi \sigma_N \geq \frac{1}{2} \tag{1}$$

with $\sigma_\phi$ and $\sigma_N$ are the standard deviations of $\phi$ and $N$. The difficulties of Dirac's approach were clearly pointed out by Susskind and Glogower [3]. Firstly, the relation (1) would imply that a well-defined number state would have a phase standard deviation greater than $2\pi$. This is a consequence of the fact that Dirac's commutator does not take account of the periodic nature of the phase. Furthermore, the exponential operator $e^{i\hat{\phi}}$ derived from this approach is not unitary and thus does not define a Hermitian operator. This is why it is often accepted that a well-behaved Hermitian phase operator does not exist [3,4]. Therefore, arguments based on the Heisenberg relation (1) cannot hold in general.

On the other hand, a traditional method of measuring phase shift is interferometry [5]. This method relies on the optical interference effect for the comparison of phases in two paths of a Mach-Zehnder interferometer. If we fix the phase delay of one path, any detected change in the output intensity of the interferometer will indicate a phase shift experienced in the other path, thus making a measurement of the phase shift. If the interferometer is properly balanced, the output intensity $I_{out}$ has the form of $I_{out} = I_{in}(1-\cos\phi)/2$ where $I_{in}$ is the intensity of the input field, and $\phi$ is the relative phase shift between the two interfering paths. If we have a well-defined amplitude in the input field, any change $\Delta I_{out}$ in the output intensity must come from the change $\Delta\phi$ in the relative phase. The sensitivity is highest when we set $\phi = \pi/2$, that is $\Delta I_{out} = I_{in}\phi/2$. Classically, there is no limit on how small the change of the intensity can be. Therefore, in principle, there is no limit on how small a phase shift

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can be measured. In quantum theory, however, the particle nature of light does not allow an infinite division of energy, thus setting a lower limit on the *change* of the output intensity. In terms of photon numbers one can write $\Delta N_{out} = N_{in}\Delta\phi/2$, where $N_{in}$ is the total input photon number and $\Delta N_{out}$ is the change in the output photon number. The minimum $\Delta N_{out}$ that is allowed by any quantum theory is simply one corresponding to the change of one quanta. Therefore, the quantum limit for phase measurement in this case is $\Delta\phi = 1/N$ with $N = N_{in}/2$ is the total number of photons in the arm of the interferometer that experiences the phase shift. However, such a semiclassical argumentation is limited to the specific scheme of interferometry for phase comparison, and to the detection scheme of intensity measurement.

In [6] it is shown through a series of arguments that, given a total average photon number $\langle N\rangle$, a fundamental limit in the precision phase measurement is set by quantum mechanics to be the so-called *Heisenberg limit*

$$\Delta\phi \geq \frac{1}{\langle N\rangle} \tag{2}$$

understood as an approximate limit at a *large* mean photon number. Now, the interest in reaching the Heisenberg limit is great because it seems to present a fundamental limit and overcomes the shot-noise limit $\Delta\phi \gtrsim 1/\sqrt{\langle N\rangle}$. Known, feasible schemes use degenerate squeezed vacuum combined with Glauber-coherent light to increase the phase sensitivity achieving sub-shot noise resolution, but do not reach the Heisenberg limit [7,8], and no practical scheme has been found that shows scaling like the Heisenberg limit for *large output intensities*.

In the following, we will establish the lack of output intensity in high precision measurments by the amount of unavoidable phase disturbance caused by any initial photon number preparation. For this purpose we consider the conditional probability of the consecutive measurment of photon number and phase. In order to specify this measurements, let us refer to the quantum formalism of [9–11], where the statistics of measurement are given by *positive operator-valued measures* [12–14]. These statistics are essentially those of the Pregg-Barnett [15,16] and also meet the experimental demands for phase measurements. Although the quantum-optical phase is still a controversial subject [17], significant progress has been made in unifying the various different formalisms of phase, and the results of different formalisms embodying the concept of phase as an observable canonical conjugate to the photon number have been shown to be physically equivalent [18]. In particular, the phase distribution associated with the Helstrom-Shapiro-Shepard probability operator measure [10,19,20] is equivalent to that of derived from Pegg-Barnett formalism [15,16] for physical states in the infinite dimensional limit [11,20,21]. The corresponding *canonical* phase distribution of $\phi$, introduced in [9,20,22], is associated to the positive operator-valued measure

$$E_\phi(X) = \frac{1}{2\pi}\sum_{n,m=0}^{\infty}\int_X e^{i(n-m)\phi}\,\mathrm{d}\phi\;\; |n\rangle\langle m| \tag{3}$$

were $X$ is some Borel set of the half-open interval $[-\pi,\pi)$. An important property of this measure is the *number-shift invariance*, which provides the key for relating canonical and measured phase distributions [14,22]. Equation (3) provides the positive operator-valued measure of an ideal measurement and may be expressed by $dE_\phi = |\phi\rangle\langle\phi|\,\mathrm{d}\phi$, where

$$|\phi\rangle = \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty} e^{in\phi}|n\rangle \tag{4}$$

These states are eigenstates of the Susskind-Glogower operators $\widehat{\cos}\phi$ and $\widehat{\sin}\phi$ and may be interpreted as phase states. We therefore see that the positive operatore-valued measure (3) is consistent with the Susskind-Glogower formalism. It can also be shown [11] that identical results are obtained using (3) as using the Pegg-Barnett formalism. In fact, that these different approaches to phase give equivalent results is a compelling reason to consider this to be the *canonical description* of phase.

With reference to (3), we now introduce the *conditional probability* of consecutive measurements of photon number and phase and apply the formalism of measurement [12–14]. For this purpose, the *precision* $\Delta\alpha \in [0,2\pi)$ of a phase measurement is defined in terms of the vicinity $A_\alpha = [\alpha - \frac{\Delta\alpha}{2}, \alpha + \frac{\Delta\alpha}{2})$ with $\alpha \in [-\pi,\pi)$. According to (3) the probability of a phase measurement event $\phi \in A_\alpha$, made on a state described by a density operator $\hat\rho$, is given by $\mathrm{tr}[\,\hat\rho\, E_\phi(A_\alpha)]$. In order to introduce the *precision* $\Delta k \in \mathbb{N}$ by which the initial photon number is measured, we bring into mind that the range of the photon number $n$ is bounded from below. Therefore, we consider the right-sided vicinity $B_k \subset \mathbb{N}$ of the photon number $k$ by $B_k = \{k, k+1, ..., k+\Delta k\}$. The probability to measure a photon number event $n \in B_k$ is given by $\mathrm{tr}[\hat\rho\, E_{\hat N}(B_k)]$, where $E_{\hat N}(B_k)$ is the value of the spectral measure $E_{\hat N}$ on the vicinity $B_k$ of $k$. For the particular case of the pure state $\hat\rho = |\psi\rangle\langle\psi|$, we have the probability $\mathrm{tr}[\,\hat\rho\, E_{\hat N}(B_k)] = ||E_{\hat N}(B_k)\,\psi||^2 = \sum_{n=k}^{k+\Delta k}|\psi_n|^2$ and $\psi_n = \langle n|\psi\rangle$ is the number-space amplitude of $\psi$.

Now, the formalism for conditional probabilities under quantum measurements is well developed [12–14]. By an initial photon number measurement, the single mode is supposed to emerge in a state according to $\hat\rho \to E_{\hat N}(B_k)\,\hat\rho\, E_{\hat N}(B_k)$. Afterwards, the number of photons is given with precision $\Delta k$. In this situation the uncertainty principle suggests that the more accurately the number is measured the greater is the perturbation of the phase of the outgoing state. The conditional probability $\mathcal{P}_{\alpha,k}(\Delta\alpha\,|\,\Delta k;\psi)$ to measure phase $\phi \in A_\alpha$, on the state transformed by the initial number measurement, is given by

$$\mathcal{P}_{\alpha,k}(\Delta\alpha\,|\,\Delta k;\psi) = \frac{||E_\phi(A_\alpha)E_{\hat N}(B_k)\,\psi||^2}{||E_{\hat N}(B_k)\,\psi||^2} \tag{5}$$

Now, our main statement is the following:

**Theorem.** Let $\Delta\alpha \in [0, 2\pi)$ and $\Delta k \in \mathbb{N}$ be fixed precisions. For every $k \in \mathbb{N}$, $\alpha \in [-\pi, \pi)$, and every Hilbert space vector $\psi$, the *least upper bound* of the conditional measurement probability is given by the inequality

$$\mathcal{P}_{\alpha,k}(\Delta\alpha\,|\,\Delta k;\psi) \le \lambda_0 \tag{6}$$

where $\lambda_0 < 1$ is the maximum eigenvalue of the trace class operator $E_{\hat N}(B_0)E_\phi(A_0)E_{\hat N}(B_0)$. Moreover we receive the inequality

$$\lambda_0 \le \xi = \frac{\Delta\alpha(\Delta k + 1)}{2\pi} \tag{7}$$

**Proof.** We consider the $(\Delta k{+}1)$-dimensional Hilbert subspace $\mathcal{H}_k = E_{\hat N}(B_k)\mathcal{H} \subset \mathcal{H}$, equipped with the canonical scalar product

$$\langle\psi|\varphi\rangle_k = \sum_{n=k}^{k+\Delta k} \psi_n^*\,\varphi_n \tag{8}$$

and norm $||\psi||_k = \sqrt{\langle\psi|\psi\rangle_k}$. For given values $k$ and $\alpha$ let us define the linear operator $\hat G_{\alpha k} : \mathcal{H}_k \to \mathcal{H}_k$ by

$$\hat G_{\alpha k} = E_{\hat N}(B_k)E_\phi(A_\alpha)E_{\hat N}(B_k) \tag{9}$$

with matrix representation[1]

$$(\hat G_{\alpha k})_{nm} = e^{i\alpha(n-m)}\,\frac{1}{\pi}\,\frac{\sin(\frac{\Delta\alpha}{2}\,(n-m))}{n-m} \tag{10}$$

Then we obtain the following representation of the measurement probability (5)

$$\mathcal{P}_{\alpha,k}(\Delta\alpha\,|\,\Delta k;\psi) \equiv \frac{\langle\psi|\,\hat G_{\alpha k}\,\psi\rangle_k}{\langle\psi|\psi\rangle_k} \tag{11}$$

On the other hand, the ordinary operator norm of $\hat G_{\alpha k}$ in $\mathcal{H}_k$ is formally given by

$$||\hat G_{\alpha k}||_k = \sup_{\psi\in\mathcal{H}_k\backslash\{0\}} \frac{|\langle\psi|\,\hat G_{\alpha k}\,\psi\rangle_k|}{\langle\psi|\psi\rangle_k} \tag{12}$$

and simply obtains the *least upper bound* of the measurement probability (5). A substantial step for the computation of $||\hat G_{\alpha k}||_k$ is given by the following:

[1] It is understood here, that when $n = m$ the expression $\frac{\sin(\frac{\Delta\alpha}{2}\,(n-m))}{n-m}$ has the value $\frac{\Delta\alpha}{2}$.

**Lemma.** For every $\alpha, k, \Delta\alpha$ and $\Delta k$, we receive the identity

$$||\hat G_{\alpha k}||_k = ||\hat G_{00}||_0 \tag{13}$$

**Proof.** We consider the shift transformation $\hat T_k$ defined by $(\hat T_k\psi)_n = \psi_{n-k}$ and the unitary transformation $\hat U_\alpha$ with $(\hat U_\alpha\psi)_n = e^{i\alpha n}\psi_n$ on the space $\mathcal{H}$. Then, by using the identities

$$\langle\psi|\,\hat G_{\alpha k}\,\psi\rangle_k = \langle\varphi_{\alpha k}|\,\hat G_{00}\,\varphi_{\alpha k}\rangle_0 \tag{14}$$

$$\langle\psi|\,\psi\rangle_k = \langle\varphi_{\alpha k}|\,\varphi_{\alpha k}\rangle_0 \tag{15}$$

with $\varphi_{\alpha k} = (\hat U_\alpha \hat T_k)^{-1}\psi$, there is the following reformulation of (12)

$$||\hat G_{\alpha k}||_k = \sup_{\varphi\in(\hat U_\alpha\hat T_k)^{-1}\mathcal{H}\backslash\{0\}} \frac{|\langle\varphi|\,\hat G_{00}\,\varphi\rangle_0|}{\langle\varphi|\varphi\rangle_0} \tag{16}$$

By using $\mathcal{H} = \hat U_\alpha \hat T_k\mathcal{H}$ the lemma is proven.

Obviously, $\hat G_{00}$ is self-adjoint, positive and the norm is equal to the maximal eigenvalue $\lambda_0$ of $\hat G_{00}$. According to the matrix representation (10), the eigenvalues of $\hat G_{00}$ satisfy the following equation for $n = 0, 1, ..., \Delta k$

$$\sum_{m=0}^{\Delta k} \frac{1}{\pi}\,\frac{\sin(\frac{\Delta\alpha}{2}\,(n-m))}{n-m}\,\psi_m^{(s)} = \lambda_s\,\psi_n^{(s)} \tag{17}$$

for eigenvectors $\psi^{(s)} \in \mathcal{H}_0$, $s = 0, 1, ..., \Delta k$. This type of eigenvalue problem has been extensively discussed in [23] (see also references therein). Both the eigenvectors and the eigenvalues are dependent on $\Delta\alpha$ and $\Delta k$. All eigenvalues are distinct, positive and may be ordered so that $\lambda_0 > \lambda_1 > ... > \lambda_{\Delta k}$. Since $\hat G_{00} = E_{\hat N}(B_0)E_\phi(A_0)E_{\hat N}(B_0)$ the first statement of the theorem is proven. Furthermore, the trace of $\hat G_{\alpha k}$ is

$$\mathrm{tr}(\hat G_{\alpha k}) = \frac{\Delta\alpha(\Delta k + 1)}{2\pi}, \tag{18}$$

corresponding to the right hand-side of (7), and $\lambda_0$ can never exceed the trace. $\Box$

In fig. 1 we see the monotonic behavior of $\lambda_0$ versus $\xi \in [0, \Delta k + 1]$, for $\Delta k = 0, 1, 2, 3$ and $\Delta k \to \infty$. The case $\Delta k = 0$ (dotted) is a straight line. For increasing values of $\Delta k$, the intermediate bounds gradually approach to the dashed line which is corresponding to the asymptotic case $\Delta k \to \infty$ with $\Delta\alpha \to 0$ and $\xi > 0$ is fixed. In this case, we introduce the transformation $q_m = \frac{m}{\Delta k + 1}$, $m = 0, 1, ..., \Delta k$, and define the increment $\delta q_m = q_{m+1} - q_m$. After substitution in (17) and a few algebraic manipulations, the eigenvalue problem approaches to the following homogeneous Fredholm integral equation of the second kind

$$\frac{1}{\pi}\int_{-1}^{1} \frac{\sin(\frac{\pi}{2}\xi(z - z'))}{z - z'}\,\varphi^{(\nu)}(z';\xi)\,\mathrm{d}z' = \tilde\lambda_\nu(\xi)\,\varphi^{(\nu)}(z;\xi) \tag{19}$$

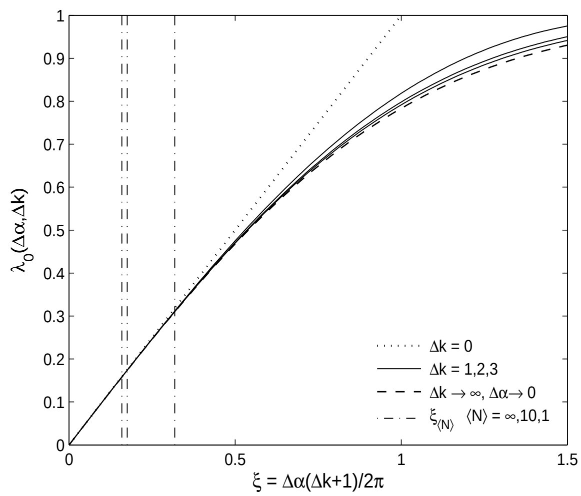


Fig. 1: Phase measuring processes with conditional probabilities (5) above the least upper bound $\lambda_0(\Delta\alpha, \Delta k)$ do not exist. The vertical lines (dashed-dotted) are corresponding to the ordinary Heisenberg limit with $\langle N\rangle = \infty, 10, 1$ (from left to right).

$|z| \leq 1$, in which the single parameter, $\xi$, appears instead of $\Delta k$ and $\Delta\alpha$ separately. From standard theory we know that (19) has solutions in $L^2([-1,1])$ only for a discrete set of eigenvalues, $\tilde\lambda_0 \geq \tilde\lambda_1 \geq ...$ and that as $\nu \to \infty$, $\lim \tilde\lambda_\nu \to 0$. It should be noted that both the $\varphi^{(\nu)}(z;\xi)$ and $\tilde\lambda_\nu(\xi)$ depend on the parameter $\xi$ and corresponding to each eigenvalue $\tilde\lambda_\nu(\xi)$ there is a unique (up to normalization) solution $\varphi^{(\nu)}(z;\xi) = S_{0\nu}(\pi\xi/2, z)$ called *angular prolate spheroidal wave function* [23, 24]. They are continuous functions of $\xi$ for $\xi \geq 0$, and are orthogonal in $(-1, 1)$. Moreover, they are complete in $L^2([-1,1])$. The corresponding eigenvalues are related to a second set of functions called *radial prolate spheroidal functions*, which differ from the angular functions only by a real scale factor [23]. Applying the notation of [25] the eigenvalues are

$$\tilde\lambda_\nu(\xi) = \xi \left[R^{(1)}_{0\nu}(\pi\xi/2, 1)\right]^2 \qquad (20)$$

with $\nu = 0, 1, 2, ...$ The properties of this spectrum for $\nu \geq 1$ is discussed in [26]. However, we are mainly interested in the properties of the largest eigenvalue $\tilde\lambda_0(\xi)$, see fig. 1 (dashed line). It is monotonically increasing and approaches 1 exponentially in $\xi$. For small values of $\xi$ there is the asymptotic behavior $\tilde\lambda_0(\xi) \sim \xi$.

A probabilistic classification of the Heisenberg limit is now straighforward. The corresponding values of $\xi$ are simply obtained by applying the measurement precisions $\Delta k = \langle N\rangle$ and $\Delta\alpha = \Delta\phi$ to the condition $\Delta\phi = 1/\langle N\rangle$. After substitution into (7) we obtain the equivalence

$$\xi_{\langle N\rangle} = \frac{1}{2\pi}\Big[1 + \frac{1}{\langle N\rangle}\Big] \qquad (21)$$

The vertical lines in fig. 1 are the corresponding dividing lines ('unit steps') of the Heisenberg limit (2) for $\langle N\rangle = \infty, 10$ and 1 (from left to right). In literature, the Heisenberg limit is understood as an approximate limit at a *large* mean photon number $\langle N\rangle$. However, already the condition $\langle N\rangle \geq 1$ implies $\xi_{\langle N\rangle} \leq 1/\pi$ and according to the second inequality of the theorem we obtain the relation $\tilde\lambda_0(\xi_{\langle N\rangle}) \leq 1/\pi$. As a consequence, the Heisenberg limit can never be reached by measurement events of probability greater than $1/\pi \approx 0.32$ when $\langle N\rangle \geq 1$.

In fact, feasible measurement schemes use degenerate squeezed vacuum combined with Glauber-coherent light to increase the phase sensitivity achieving sub-shot noise resolution, but do not reach the Heisenberg limit [7, 8], and no practical scheme has been found that shows scaling like the Heisenberg limit for large intensities. As a matter of fact, for the constitution of a measurement apparatus with higher measurement probabilities, higher values of $\tilde\lambda_0(\xi)$ are necessary. For instance, to attain measurement probabilities of at least 0.78, a bound of $\tilde\lambda_0(\xi) \geq 0.78$ is a necessary condition. In this case, we obtain the necessary condition $\xi \geq 1$, or equivalently, $\Delta\phi \geq 2\pi/\langle N\rangle$ for large mean photon numbers.

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